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CENTRALLY SYMMETRIC CONVEX BODIES AND DISTRIBUTIONS

BY
WOLFGANG WEIL

ABSTRACT
To each centrally symmetric convex body is assigned a distribution on the sphere. As applications, geometric formulas and a characterization of zonoids are obtained.

Introduction

In the theory of convex bodies (nonvoid, compact, convex sets) in Euclidean d-space $E^d (d \geqq 2)$, one often is interested in bodies which are "composed" of certain "simple" ones. For the purpose of our investigations, "composition" means vector addition and the "simple" bodies are the one-dimensional compact, convex sets, the line segments.

Finite sums of line segments are convex polytopes which are characterized by a strong property of symmetry. They, as well as all their faces (equivalently two-dimensional faces), are centrally symmetric. Such polytopes, now called zonotopes, were noticed first, for $d = 3$, by the Russian crystallographer Fedorov in 1885. The support function H_P of a zonotope P is the sum of the support functions of the line segments s_i, whose sum is P. If the origin is the centre of symmetry of P, we can assume the same for s_i. Because $|\langle x, u \rangle|$, $u \in E^d$ (where $\langle \cdot, \cdot \rangle$ is the scalar product) is the support function of the line segment with endpoints x and $-x$, we may write

$$H_P(u) = \sum_{i=1}^{n} |\langle x_i, u \rangle| \cdot \rho_i, \; u \in E^d,$$

with unit vectors $x_1, \cdots, x_n$ and $\rho_i \geqq 0$, $i = 1, \cdots, n$.

This equation suggests the study of convex bodies K, whose support functions are representable by

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$$H_K(u) = \int_\Omega |\langle x, u\rangle| \rho(dx), u \in E^d,$$

with a (nonnegative) Borel measure ρ on the unit sphere Ω in E^d. These bodies and their translates are called zonoids. They could be considered as continuous positive combinations of line segments, and the set of zonoids is exactly the closure of the set of zonotopes in the usual Hausdorff metric. The study of zonotopes and zonoids involves a lot of interesting problems; a survey of references can be found in [9].

In his book "Kreis und Kugel" [2, pp. 154–155], Blaschke mentioned a wider class of convex bodies. He showed that any positively homogeneous, even function f on E^3, satisfying some smoothness condition, can be written in the form

$$f(u) = \int_\Omega |\langle x, u\rangle| \rho(dx), u \in E^3,$$

with a suitable, signed Borel measure ρ on Ω. Moreover, if ρ is assumed to be even, then it is uniquely determined by f. We call the translates of convex bodies, whose support functions have this representation, generalized zonoids; they are dense in the set of all centrally symmetric bodies. Schneider [5], [6] made Blaschke's smoothness condition precise for arbitrary d and gave examples of centrally symmetric convex bodies which are not generalized zonoids, and of generalized zonoids which are not zonoids.

The correspondence between K and ρ for a dense subset engenders the feeling that there is a correspondence between general, centrally symmetric convex bodies and certain mathematical objects on Ω. The sequence

measure of finite support $\to$ measure $\to$ signed measure $\to$?

suggests that these "objects" are distributions. The existence of such a correspondence between centrally symmetric convex bodies and distributions on Ω is not difficult to derive from Blaschke's result. This is not only of a purely theoretical interest. For we have pointed out in [9] the connection between the signed measures ρ and the geometric properties of the generalized zonoids; hence it may be of interest, whether these results can be transformed to arbitrary, centrally symmetric, convex bodies. It is the aim of this paper to study the indicated connection between centrally symmetric, convex bodies and distributions on Ω and to prove some geometric formulas.

As a consequence, we get, by a simple argument from distribution theory, a characterization of zonoids by inequalities of mixed volumes which by itself may be a justification of these investigations.

The setup of the paper is as follows. Section 1 contains a summary of needed notations and results concerning convex bodies. In Section 2 we collect the notations and properties of distributions on Ω. In Section 3 we assign to each convex body, centrally symmetric with respect to the origin, an even distribution on Ω. Then we study the domain of definition of these distributions, and we find an upper bound for their order. Section 4 contains the formulas for the support function, the mixed volumes, and the mixed surface area measures. The last section deals with our criterion for zonoids.

1. Convex bodies

In the sequel, we list the most usual notations and facts about convex bodies, as found in a concentrated form in Bonnesen and Fenchel [3] or in Busemann [4], especially the part about surface area measures.

Let E^d ($d \geqq 2$) be the d-dimensional Euclidean space. Elements $x, y \in E^d$ are d-tuples $x = (x^1, \cdots, x^d)$, $y = (y^1, \cdots, y^d)$ with real numbers x^i, y^i. The scalar product is $\langle x, y\rangle = x^1y^1 + \cdots + x^dy^d$ and the norm $\| x \| = \langle x, x\rangle^{1/2}$. $B = \{x \in E^d \mid \| x \| \leqq 1\}$ is the unit ball and $\Omega = \{x \in E^d \mid \| x \| = 1\}$ the unit sphere. We supply Ω with the σ-algebra $\mathcal{B}$ of Borel subsets. A measure ρ on $(\Omega, \mathcal{B})$ is, in our sense, a nonnegative, finite, σ-additive set function on $\mathcal{B}$. A signed measure is a difference of measures. A (signed) measure ρ is even if $\rho(A) = \rho(\{x \mid -x \in A\})$ for all $A \in \mathcal{B}$. Let $\mathcal{M}(\Omega)$ be the set of all even, signed measures on $(\Omega, \mathcal{B})$ and $\mathcal{M}_+(\Omega) = \{\rho \in \mathcal{M}(\Omega) \mid \rho \geqq 0\}$.

A convex body K is a nonvoid, compact, convex subset of E^d. K is a polytope if it is the convex hull of finitely many points. K is centrally symmetric if there is a point $x_0 \in K$, the centre, such that $y \in K$ implies $2x_0 - y \in K$ for all $y \in E^d$. Let $\mathcal{K}$ be the set of all convex bodies $K \subset E^d$ which are centrally symmetric and have the origin as centre. In what follows, we are mostly concerned with convex bodies which are elements of $\mathcal{K}$. The (vector) sum of two convex bodies K, L is defined as

$$K + L = \{x + y \mid x \in K, y \in L\}.$$

The multiple αK of K, where $\alpha \geqq 0$, is defined as

$$\alpha K = \{\alpha x \mid x \in K\}.$$

For convex bodies K, L and $\alpha, \beta \geqq 0$, $\alpha K + \beta L$ is a convex body, and if $K, L \in \mathcal{K}$, then $\alpha K + \beta L \in \mathcal{K}$. Thus, $\mathcal{K}$ is a convex cone. The Hausdorff metric d is defined by

$$d(K, L) = \inf\{\varepsilon > 0 \mid K \subset L + \varepsilon B, L \subset K + \varepsilon B\}.$$

$\mathcal{K}$ is closed in the topology generated by d. The support function H_K of a convex body K is

$$H_K(u) = \sup_{x \in K} \langle x, u \rangle, u \in E^d.$$

H_K is continuous, positively homogeneous, and convex. $K \in \mathcal{K}$ if and only if H_K is even, that is $H_K(-x) = H_K(x)$ for all $x \in E^d$. If $\mathscr{C}(\Omega)$ denotes the Banach space of even, continuous functions f on Ω with the norm

$$\|f\| = \sup_{x \in \Omega} |f(x)|,$$

then the restrictions of even support functions to Ω form a closed convex cone $\mathscr{H}(\Omega)$ in $\mathscr{C}(\Omega)$. By the correspondence between convex bodies and support functions, $\mathcal{K}$ and $\mathscr{H}(\Omega)$ are isomorphic (in the algebraical and topological sense).

The volume $V(K)$ of a convex body K is the Lebesgue measure of K. Approximating a convex body K by polytopes, one can prove that the volume of a linear combination of convex bodies $\alpha_1 K_1 + \cdots + \alpha_n K_n$, $\alpha_i \geqq 0$, is a polynomial in α_i, $i = 1, \cdots, n$, where the coefficient of $\alpha_{i_1} \times \cdots \times \alpha_{i_d}$ is symmetric and depends only on $K_{i_1}, \cdots, K_{i_d}$:

$$V(\alpha_1 K_1 + \cdots + \alpha_n K_n) = \sum_{\substack{i_j = 1, \cdots, n \\ j = 1, \cdots, d}} \alpha_{i_1} \times \cdots \times \alpha_{i_d} V(K_{i_1}, \cdots, K_{i_d}).$$

$V(K_1, \cdots, K_d)$ is called the mixed volume of $K_1, \cdots, K_d$. It is multilinear, continuous, nonnegative, and monotone with respect to the inclusion order. The special mixed volumes

$$W_j(K) = V(\underbrace{K, \cdots, K}_{d-j}, \underbrace{B, \cdots, B}_{j}), \quad j = 0, 1, \cdots, d$$

are called quermassintegrals. $W_0(K)$ is $V(K)$, $d \cdot W_1(K)$ is the surface area of K, $W_d(K) = W_0(B)$ is $V(B)$. We use the abbreviation $\kappa_d = V(B)$.

Now, for fixed $K_1, \cdots, K_{d-1} \in \mathcal{K}$, $V(\cdot, K_1, \cdots, K_{d-1})$ is a continuous, monotone, linear functional on $\mathcal{K}$ and, by the isomorphism of $\mathcal{K}$ and $\mathscr{H}(\Omega)$, on $\mathscr{H}(\Omega)$. The vector space $\mathscr{L}(\Omega)$, generated by $\mathscr{H}(\Omega)$, is dense in $\mathscr{C}(\Omega)$. One can show that $V(\cdot, K_1, \cdots, K_{d-1})$ is continuous on $\mathscr{L}(\Omega)$; hence it can be extended to a unique, continuous, monotone, linear functional on $\mathscr{C}(\Omega)$. Such a functional corresponds to a measure $\mu \in \mathscr{M}_+(\Omega)$. We write $d \cdot \mu = \mu(K_1, \cdots, K_{d-1}; \cdot)$ and have

$$d \cdot \mu(f) = \int_\Omega f(x)\mu(K_1, \cdots, K_{d-1}; dx) \quad \text{for all} \quad f \in \mathscr{C}(\Omega).$$

In particular, we get

$$V(K_1, \cdots, K_d) = d^{-1} \int_\Omega H_{K_d}(x)\mu(K_1, \cdots, K_{d-1}; dx)$$

for all $K_1, \cdots, K_d \in \mathscr{K}$. $\mu(K_1, \cdots, K_{d-1}; \cdot)$ is called the mixed surface area measure of $K_1, \cdots, K_{d-1}$, and the measure

$$\mu_j(K; \cdot) = \mu(\underbrace{K, \cdots, K}_{j}, \underbrace{B, \cdots, B}_{d-1-j}; \cdot), \quad j = 0, 1, \cdots, d-1,$$

is called the j-th order surface area measure of $K \in \mathscr{K}$. Because $\mu_{d-1}(K; \Omega) = d \cdot W_1(K)$ is the surface area of K, the notation is reasonable. $\mu_0(K; \cdot) = \mu_{d-1}(B; \cdot)$, the $(d-1)$-th order surface area measure of B, is the Lebesgue measure on $(\Omega, \mathscr{B})$. We denote it by λ.

A line segment s is a one-dimensional convex body, $s = \{\alpha y + (1-\alpha)y' \mid 0 \leqq \alpha \leqq 1\}$ with $y, y' \in E^d$. s belongs to $\mathscr{K}$ if $y = -y'$. A zonotope $P \in \mathscr{K}$ is a sum $P = s_1 + \cdots + s_n$ with line segments $s_i \in \mathscr{K}$. We have

$$H_P(u) = \int_\Omega |\langle x, u \rangle| \rho(dx), u \in E^d,$$

for a unique measure $\rho \in \mathscr{M}_+(\Omega)$ supported by a finite number of points. A zonoid $K \in \mathscr{K}$ is a limit of zonotopes. Again we have

$$H_K(u) = \int_\Omega |\langle x, u \rangle| \rho(dx), u \in E^d,$$

for a unique $\rho \in \mathscr{M}_+(\Omega)$. A generalized zonoid $K \in \mathscr{K}$ is a convex body such that there exist zonoids $K_1, K_2 \in \mathscr{K}$ with $K_1 = K + K_2$. Equivalently,

$$H_K(u) = \int_\Omega |\langle x, u \rangle| \rho(dx), u \in E^d,$$

for a unique signed measure $\rho \in \mathscr{M}(\Omega)$.

2. Distributions on the sphere

Distributions on E^d are continuous linear functionals on the locally convex space of infinitely differentiable functions on E^d with compact support. An analogous definition and development of the theory is possible if E^d is replaced by a differentiable manifold (see Schwartz [7, pp. 31–32]). Because of the compactness of Ω, the structure of distributions on Ω is simpler than in the general theory. On the other hand, methods which depend on the vector space structure of E^d, such as convolution, are not immediately transferable to Ω. A development of such techniques for distributions on Ω as well as a first application of distribution theory to convex bodies is given by Berg [1]. We introduce now the function spaces which will be used and the most important facts about distributions which are presumed in the next chapters.

For simplicity, functions on Ω are regarded just as functions on E^d, positively homogeneous of degree one. Differentiability on Ω then corresponds to differentiability on $E^d\backslash\{0\}$. This clarifies the meaning of partial derivatives in what follows.

For $m \in \{0,1,2,\cdots\}$ let $\tilde{\mathscr{D}}_m(\Omega)$ be the vector space of all m-times continuously differentiable real functions f on Ω. If $p = (p_1,\cdots,p_d)$, $p_i \in \{0,1,2,\cdots\}$, is a multi-index with $|p| = p_1 + \cdots + p_d$, we set

$$\partial^p f(x) = \frac{\partial^{|p|} f}{\partial(x^1)^{p_1}\cdots\partial(x^d)^{p_d}}(x).$$

We equip $\tilde{\mathscr{D}}_m(\Omega)$ with the normable topology generated by the finite family of semi-norms

$$\{q_p \mid q_p(f) = \max_{x\in\Omega} |\partial^p f(x)|,\ |p| \leqq m\}.$$

We set $\tilde{\mathscr{D}}(\Omega) = \bigcap_{m=0}^{\infty} \tilde{\mathscr{D}}_m(\Omega)$, where $\tilde{\mathscr{D}}(\Omega)$ carries the projective topology. The spaces $\mathscr{D}(\Omega)$, $\mathscr{D}_m(\Omega)$ are defined as those subspaces of $\tilde{\mathscr{D}}(\Omega)$, $\tilde{\mathscr{D}}_m(\Omega)$ which consist of all even functions. We have $\mathscr{D}_0(\Omega) = \mathscr{C}(\Omega)$. Marking the dual space by a prime, we call the elements of $\tilde{\mathscr{D}}'(\Omega)$ distributions on Ω and the elements of $\tilde{\mathscr{D}}'_m(\Omega)$ distributions of order $\leqq m$. The compactness of Ω implies that all distributions on Ω are of finite order, that is

$$\tilde{\mathscr{D}}'(\Omega) = \bigcup_{m=0}^{\infty} \tilde{\mathscr{D}}'_m(\Omega).$$

For a function f on Ω, let f^* be defined by $f^*(x) = f(-x), x \in \Omega$. We call a distribution $T \in \tilde{\mathscr{D}}'(\Omega)$ even if $T(f) = T(f^*)$ for all $f \in \tilde{\mathscr{D}}(\Omega)$. The dual spaces $\mathscr{D}'(\Omega), \mathscr{D}'_m(\Omega)$ are identified with the subspaces of even distributions in $\tilde{\mathscr{D}}'(\Omega), \tilde{\mathscr{D}}'_m(\Omega)$. In particular we have $\mathscr{D}'_0(\Omega) = \mathscr{M}(\Omega)$. Hence, for $\rho \in \mathscr{M}(\Omega)$ and $f \in \mathscr{C}(\Omega)$, we shall sometimes use $\rho(f)$ instead of $\int_\Omega f(x)\rho(dx)$. If in the sequel a topology on $\mathscr{D}'(\Omega)$ or $\mathscr{D}'_m(\Omega)$ is needed, we always take the strong topology unless otherwise stated.

For natural numbers n, let

$$\Omega^n = \underbrace{\Omega \times \cdots \times \Omega}_{n}$$

be the Cartesian product and let

$$\mathscr{F}^n(\Omega) = \underbrace{\mathscr{F}(\Omega) \otimes \cdots \otimes \mathscr{F}(\Omega)}_{n}$$

be the tensor product, where $\mathscr{F}(\Omega)$ is any space of functions on Ω. The tensor product of distributions $T_1, \cdots, T_n$ is denoted by $T_1 \otimes \cdots \otimes T_n$ and

$$T^n = \underbrace{T \otimes \cdots \otimes T.}_{n}$$

We have $T_1 \otimes \cdots \otimes T_n \in \mathscr{D}'(\Omega^n)$, if each $T_i \in \mathscr{D}'(\Omega)$. Distributions in $\mathscr{D}'_0(\Omega)$ are signed measures μ. The tensor product $\mu_1 \otimes \cdots \otimes \mu_n$ is then the usual (signed) product measure.

Finally, we will introduce a function space on Ω which is especially important in the context of our investivations. Let $\mathscr{E}(\Omega)$ be the space of all functions f on Ω such that there is a signed measure $\rho_f \in \mathscr{M}(\Omega)$ with

$$f(u) = \int_\Omega |\langle x, u \rangle| \, \rho_f(dx), \, u \in \Omega.$$

We call ρ_f the generating signed measure, according to the similar notation for support functions f (see [9]). If $K \in \mathscr{K}$ is a generalized zonoid, we shall write ρ_K instead of ρ_{H_K}. Blaschke's result gives us

$$\mathscr{D}(\Omega) \subset \mathscr{E}(\Omega) \subset \mathscr{C}(\Omega).$$

Schneider's calculations [5] imply moreover

$$\mathscr{D}_{d+2}(\Omega) \subset \mathscr{E}(\Omega) \quad \text{for} \quad d \quad \text{even},$$

$$\mathscr{D}_{d+3}(\Omega) \subset \mathscr{E}(\Omega) \quad \text{for} \quad d \quad \text{odd}.$$

If $\mathscr{E}(\Omega)$ is equipped with the topology of weak convergence of the generating signed measures, then $\mathscr{E}(\Omega)$ is isomorphic to $\mathscr{M}(\Omega)$, carrying the weak topology, and the embeddings just mentioned are all continuous.

3. The correspondence between centrally symmetric convex bodies and distributions

We assign now, to each convex body $K \in \mathcal{K}$ a distribution $T_K \in \mathcal{D}'(\Omega)$. We shall see that, by this representation, the cone $\mathcal{K}$ becomes isomorphic to a cone of distributions. The proof of this isomorphism and the investigation of the occurring distributions are the contents of this chapter.

THEOREM 3.1. *For each $K \in \mathcal{K}$, the real functional T_K, defined on $\mathcal{D}(\Omega)$ by*

$$T_K(f) = \rho_f(H_K),$$

is in $\mathcal{D}'(\Omega)$. We call T_K the generating distribution of K.

PROOF. $\mathcal{E}(\Omega) \cap \mathcal{H}(\Omega)$ is dense in $\mathcal{H}(\Omega)$ (in the topology of $\mathcal{H}(\Omega)$). Hence, for $K \in \mathcal{K}$, there exist $K_i \in \mathcal{K}$, $i = 1, 2, \cdots$, converging to K and with $H_{K_i} \in \mathcal{E}(\Omega)$. Using the fact $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ and Fubini's theorem, we get for the generating signed measures ρ_{K_i} and for all $f \in \mathcal{D}(\Omega)$:

$$\rho_{K_i}(f) = \int_\Omega f(u)\rho_{K_i}(du) = \int_\Omega \int_\Omega |\langle x, u\rangle| \rho_f(dx)\rho_{K_i}(du)$$

$$= \int_\Omega \int_\Omega |\langle x, u\rangle| \rho_{K_i}(du)\rho_f(dx) = \int_\Omega H_{K_i}(x)\rho_f(dx)$$

$$= \rho_f(H_{K_i}).$$

For $i \to \infty$, we have $H_{K_i} \to H_K$ in $\mathcal{C}(\Omega)$, therefore $\rho_f(H_{K_i}) \to \rho_f(H_K)$. This implies

$$\rho_{K_i}(f) \to T_K(f), \quad \text{for all} \quad f \in \mathcal{D}(\Omega).$$

Then T_K is the limit (in the strong topology) of the distributions $\rho_{K_i} \in \mathcal{D}'(\Omega)$ and hence a distribution in $\mathcal{D}'(\Omega)$ (see [7, p. 74]). Q.E.D.

We set $\mathcal{T}: K \mapsto T_K, K \in \mathcal{K}$, and $\mathcal{T}\mathcal{K} = \{T_K \mid K \in \mathcal{K}\}$.

THEOREM 3.2. *$\mathcal{T}\mathcal{K}$ is a closed convex cone in $\mathcal{D}'(\Omega)$.*
$\mathcal{T}: \mathcal{K} \to \mathcal{T}\mathcal{K}$ is an algebraical and topological isomorphism.

PROOF. $\mathcal{T}$ is obviously a linear mapping on $\mathcal{K}$:

$$T_{\alpha K + \beta K'}(f) = \rho_f(H_{\alpha K + \beta K'}) = \alpha\rho_f(H_K) + \beta\rho_f(H_{K'})$$

$$= \alpha T_K(f) + \beta T_{K'}(f), f \in \mathcal{D}(\Omega), \alpha, \beta \geqq 0, K, K' \in \mathcal{K}.$$

Therefore, the image $\mathcal{T}\mathcal{K}$ of the convex cone $\mathcal{K}$ is a convex cone. Next $\mathcal{T}$ is one-to-one because $T_K = T_{K'}$ implies

$$\rho_f(H_K) = \rho_f(H_{K'}) \text{ for all } f \in \mathcal{D}(\Omega),$$

thus,

$$\rho(H_K) = \rho(H_{K'}) \text{ for all } \rho \in \mathcal{M}(\Omega).$$

This yields $H_K = H_{K'}$, and $K = K'$.

$\mathcal{T}$ is continuous because $K_i \to K$, K_i, $K \in \mathcal{K}$, implies $H_{K_i} \to H_K$ and, therefore, $\rho_f(H_{K_i}) \to \rho_f(H_K)$, $f \in \mathcal{D}(\Omega)$. Hence, $T_{K_i} \to T_K$ in $\mathcal{D}'(\Omega)$. Finally, we show that the inverse $\mathcal{T}^{-1}$: $T_K \mapsto K$ is continuous. Suppose $T_K \in \mathcal{T}\mathcal{K}$ is the limit of a sequence $T_{K_i} \in \mathcal{T}\mathcal{K}$, $i = 1, 2, \cdots$. Hence, $T_{K_i}(f) \to T_K(f)$ for all $f \in \mathcal{D}(\Omega)$. Using the definition of $T_{K_i}(f)$, we get the convergence of $\rho_f(H_{K_i})$, as $i \to \infty$, for all $f \in \mathcal{D}(\Omega)$. Because

$$f(u) = \int_\Omega |\langle x, u\rangle| \, h(x)\lambda(dx), \; u \in \Omega,$$

is in $\mathcal{D}(\Omega)$, provided $h \in \mathcal{D}(\Omega)$, we derive the convergence of

$$\int_\Omega H_{K_i}(u) h(u) \lambda(du)$$

for all $h \in \mathcal{D}(\Omega)$. Taking $x_i \in K_i$ and $h \equiv 1$, we have

$$0 \leqq \int_\Omega |\langle x_i, u\rangle| \, \lambda(du) \leqq \int_\Omega H_{K_i}(u)\lambda(du) \leqq M.$$

Because of

$$\int_\Omega |\langle x_i, u\rangle| \, \lambda(du) = 2\kappa_{d-1} \| x_i \|,$$

the x_i, and hence the bodies K_i, are bounded. Blaschke's selection theorem [3, p. 34] implies the existence of a converging subsequence K_{i_j}. If K' is the limit of K_{i_j}, $j \to \infty$, we get

$$\int_\Omega H_{K'}(u) h(u) \lambda(du) = \int_\Omega H_K(u) h(u) \lambda(du)$$

for all $h \in \mathcal{D}(\Omega)$. Hence, we have $H_{K'} = H_K$, which implies $K' = K$. Thus, we have shown, that K_i converges to K and $\mathcal{T}^{-1}$ is continuous.

As a consequence, $\mathcal{T}\mathcal{K}$ is closed, and the theorem is proved. Q.E.D.

The following theorem gives us some more information about $\mathcal{T}\mathcal{K}$. We show that all generating distributions are continuous with respect to the topology defined on $\mathcal{D}(\Omega)$ by $\mathcal{E}(\Omega)$. Hence, they can be extended to elements of $\hat{\mathcal{E}}'(\Omega)$, the dual of the completion of $\mathcal{E}(\Omega)$. As a consequence, we get an upper bound for the order of T_K, $K \in \mathcal{K}$.

THEOREM 3.3. *We have* $\mathcal{T}\mathcal{K} \subset \hat{\mathcal{E}}'(\Omega)$.

PROOF. For $f \in \mathcal{E}(\Omega)$ and $K \in \mathcal{K}$, we extend T_K, setting $T_K(f) = \rho_f(H_K)$. Because convergence in $\mathcal{E}(\Omega)$ was defined as the weak convergence of the generating signed measures ρ_f, T_K is continuous on $\mathcal{E}(\Omega)$, and hence on $\hat{\mathcal{E}}(\Omega)$.
Q.E.D.

.COROLLARY 3.4. *We have* $\mathcal{T}\mathcal{K} \subset \mathcal{D}'_{d+2}(\Omega)$ *for d even and* $\mathcal{T}\mathcal{K} \subset \mathcal{D}'_{d+3}(\Omega)$ *for d odd.*

PROOF. The assertions follow immediately from the remark at the end of the last chapter. Q.E.D.

The numbers $d+2$ (resp. $d+3$) are surely not optimal, for if $f \in \mathcal{D}_{d+2}(\Omega)$ (resp. $\mathcal{D}_{d+3}(\Omega)$) then ρ_f has a continuous density with respect to the Lebesgue measure λ. The characterization of $\mathcal{E}(\Omega)$, and hence $\hat{\mathcal{E}}'(\Omega)$, is equivalent to the characterization of generalized zonoids (see [9]) which is unknown.

In the next chapter we shall use the tensor product $T_{K_1} \otimes \cdots \otimes T_{K_n}$ for natural n and $T_{K_j} \in \mathcal{T}\mathcal{K}$, $j = 1, \cdots, n$, which is a continuous linear form on the completion $(\hat{\mathcal{E}}^n(\Omega))^\wedge$ of the space $\hat{\mathcal{E}}^n(\Omega)$, supplied with the projective topology. We have $\mathcal{D}(\Omega^n) \subset (\hat{\mathcal{E}}^n(\Omega))^\wedge \subset \mathcal{C}(\Omega^n)$. Instead of investigating whether a function $f \in \mathcal{C}(\Omega^n)$ belongs to $(\hat{\mathcal{E}}^n(\Omega))^\wedge$, we use a direct definition of the existence and value of the expression $T_{K_1} \otimes \cdots \otimes T_{K_n}(f), f \in \mathcal{C}(\Omega^n)$. In what follows, $T_{K_1} \otimes \cdots \otimes T_{K_n}(f)$ is defined as the limit of $\rho_{K_1^i} \otimes \cdots \otimes \rho_{K_n^i}(f)$, as $i \to \infty$, whenever this limit exists for every choice of generalized zonoids $K_j^i \in \mathcal{K}$, converging to K_j, as $i \to \infty$, $j = 1, \cdots, n$.

It could be easily seen that, in this sense, $T_K^n(f)$ and $T_K^m \otimes \lambda^{n-m}(f)$ exist for $K \in \mathcal{K}$, if and only if $\rho_{K^i}^n(f)$ and $\rho_{K^i}^m \otimes \lambda^{n-m}(f)$ converge for each choice of generalized zonoids $K^i \in \mathcal{K}$, $K^i \to K$ as $i \to \infty$.

4. Geometric formulas

We begin this chapter with the formula for the support function.

THEOREM 4.1. *For* $K \in \mathcal{K}$ *and* $u \in \Omega$ *we have*

$$H_K(u) = T_K(|\langle u, \cdot \rangle|).$$

PROOF. We have $|\langle u, \cdot\rangle| \in \mathscr{E}(\Omega)$, hence, in view of Theorem 3.3, $T_K(|\langle u, \cdot\rangle|)$ exists and equals $\rho_{|\langle u, \cdot\rangle|}(H_K) = \frac{1}{2}(H_K(u) + H_K(-u)) = H_K(u)$. Q.E.D.

The following formulas treat the mixed volumes, the quermassintegrals, and the mixed surface area measures. All results are easily derived from the corresponding formulas for generalized zonoids which are obtained in [9].

We have to introduce the function $D_n \in \mathscr{C}(\Omega^n)$, $n = 1, \cdots, d$. For $x_1, \cdots, x_n \in \Omega$, $D_n(x_1, \cdots, x_n)$ is the absolute value of the determinant of $x_1, \cdots, x_n$ with respect to a n-dimensional subspace of E^d which contains $x_1, \cdots, x_n$. We define the mapping $R: M^{d-1} \to \Omega$, where M^{d-1} is the set of linearly independent $(d-1)$-tuples in Ω^{d-1}, by taking a fixed orientation in E^d and requiring:

(1) $R(x_1, \cdots, x_{d-1})$ is orthogonal to each x_j, $j = 1, \cdots, d-1$.

(2) $(R(x_1, \cdots, x_{d-1}), x_1, \cdots, x_{d-1})$ is positively oriented.

The mapping $S: \mathscr{C}(\Omega) \to \mathscr{C}(\Omega^{d-1})$ is then defined by

$$S(f)(x_1, \cdots, x_{d-1}) = \begin{cases} f(R(x_1, \cdots, x_{d-1}))D_{d-1}(x_1, \cdots, x_{d-1}), \\ \qquad\qquad \text{if } (x_1, \cdots, x_{d-1}) \in M^{d-1}, \\ 0 \text{ otherwise.} \end{cases}$$

THEOREM 4.2. *For $K_1, \cdots, K_d \in \mathscr{K}$, we have*

$$V(K_1, \cdots, K_d) = \frac{2^d}{d!}(T_{K_1} \otimes \cdots \otimes T_{K_d})(D_d).$$

PROOF. For each $j \in \{1, \cdots, d\}$ let $\{K_j^i\}_{i=1,2,\cdots}$ be a sequence of generalized zonoids converging to K_j. We have

$$V(K_1^i, \cdots, K_d^i) \xrightarrow[i \to \infty]{} V(K_1, \cdots, K_d)$$

and ([9], theorem 4)

$$V(K_1^i, \cdots, K_d^i) = \frac{2^d}{d!}\int_\Omega \cdots \int_\Omega D_d(x_1, \cdots, x_d)\rho_{K_1^i}(dx_1) \cdots \rho_{K_d^i}(dx_d)$$

$$= \frac{2^d}{d!}(\rho_{K_1^i} \otimes \cdots \otimes \rho_{K_d^i})(D_d).$$

Thus $2^d/d!(T_{K_1} \otimes \cdots \otimes T_{K_d})(D_d)$ exists and equals $V(K_1, \cdots, K_d)$. Q.E.D.

In the same manner, we obtain the following theorems from [9], theorems 6 and 3.

THEOREM 4.3. *For $K \in \mathcal{K}$ and $j \in \{0, \cdots, d-1\}$ we have*

$$W_j(K) = \frac{2^{d-j} \cdot j! \kappa_j}{d!} T_K^{d-j}(D_{d-j}).$$

THEOREM 4.4. *For $K_1, \cdots, K_{d-1} \in \mathcal{K}$, we have for all $f \in \mathcal{C}(\Omega)$*

$$\int_\Omega f(x)\mu(K_1, \cdots, K_{d-1}; dx) = \frac{2^d}{(d-1)!}(T_{K_1} \otimes \cdots \otimes T_{K_{d-1}})(S(f)).$$

In particular, for $K \in \mathcal{K}$, $j \in \{1, \cdots, d-1\}$, and $f \in \mathcal{C}(\Omega)$, we have

$$\int_\Omega f(x)\mu_j(K; dx) = \frac{2^{j+1}}{(d-1)!\kappa_{d-1}^{d-1-j}}(T_K^j \otimes \lambda^{d-j-1})(S(f)).$$

We use Theorem 4.4 to show that the behavior of the support function H_K of a body $K \in \mathcal{K}$ on a set $A \subset \Omega$ depends only on the restriction of T_K to a set orthogonal to A. In particular, for a set $A \subset \Omega$, let $A^\perp$ be defined by

$$A^\perp = \{x \in \Omega \mid \langle x, y \rangle = 0 \text{ for some } y \in A\}.$$

The restriction $T|_A$ of a distribution $T \in \mathcal{D}'(\Omega)$ to an open set $A \subset \Omega$ is the unique distribution in $\mathcal{D}'(\Omega)$ which satisfies: $T|_A(f) = T(f)$ for all $f \in \mathcal{D}(\Omega)$ with support in A.

THEOREM 4.5. *For $K \in \mathcal{K}$ and an open, connected subset $A \subset \Omega$, there is a vector $v \in E^d$ such that*

$$H_K(u) = T_K|_{A^\perp}(|\langle u, \cdot \rangle|) + \langle u, v \rangle \quad \textit{for all} \quad u \in A.$$

PROOF. By theorem 5 in [8], on A, H_K is determined up to a translation by the values

$$\int_\Omega f(x)\mu(\underbrace{K, \cdots, K}_{j}, K_{j+1}, \cdots, K_{d-1}; dx)$$

for all $j \in \{1, \cdots, d-1\}$, all convex bodies $K_{j+1}, \cdots, K_{d-1}$, and all continuous functions f on Ω with support in A. In view of the central symmetry of K, it suffices to know these values for $K_{j+1}, \cdots, K_{d-1} \in \mathcal{K}$ and all functions $f \in \mathcal{C}(\Omega)$ with support in $A \cup \{-x \mid x \in A\}$. We then use Theorem 4.4 and get

$$\int_\Omega f(x)\mu(\underbrace{K, \cdots, K}_{j}, K_{j+1}, \cdots, K_{d-1}; dx)$$

$$= \frac{2^d}{(d-1)!}(T_K^j \otimes T_{K_{j+1}} \otimes \cdots \otimes T_{K_{d-1}})(S(f)).$$

But $S(f)$ has support in $(A^{\perp})^{d-1}$ and, thus, we can restrict the distributions $T_K, T_{K_{j+1}}, \cdots, T_{K_{d-1}}$ to $A^{\perp}$ without changing the value of the integral. This completes the proof of the theorem. Q.E.D.

5. Characterization of zonoids

A convex body $K \in \mathcal{K}$ is a zonoid if and only if $T_K \in \mathcal{M}_+(\Omega)$. The latter is equivalent to $T_K(f) \geqq 0$ for all $f \geqq 0$, $f \in \mathcal{D}(\Omega)$ (see [7, p. 29]). The transposition of this fact into terms of mixed volumes gives our criterion for zonoids.

For a $K \in \mathcal{K}$ and $u \in \Omega$ we denote by $v(K, u)$ the $(d-1)$-dimensional volume of the projection of K onto a hyperplane orthogonal to u.

THEOREM 5.1. *A body $K \in \mathcal{K}$ is a zonoid if and only if*

$$V(K, L_1, \cdots, L_1) \leqq V(K, L_2, \cdots, L_2)$$

for all $L_1, L_2 \in \mathcal{K}$ which fulfill $v(L_1, u) \leqq v(L_2, u)$ for all $u \in \Omega$.

PROOF. Suppose, first, K is a zonoid:

$$H_K(u) = \int_\Omega |\langle x, u\rangle| \rho_K(dx), \ \rho_K \in \mathcal{M}_+(\Omega).$$

Then, for $i = 1, 2$,

$$\begin{aligned} V(K, L_i, \cdots, L_i) &= \int_\Omega H_K(u)\mu_{n-1}(L_i; du) \\ &= \int_\Omega \int_\Omega |\langle x, u\rangle| \rho_K(dx)\mu_{n-1}(L_i; du) \\ &= \int_\Omega \int_\Omega |\langle x, u\rangle| \mu_{n-1}(L_i; du)\rho_K(dx). \end{aligned}$$

Because of $v(L_i, u) = \frac{1}{2}\int_\Omega |\langle x, u\rangle| \mu_{n-1}(L_i; du)$, $i = 1, 2$, (see [5, p. 73]), we deduce from $v(L_1, u) \leqq v(L_2, u)$ for all $u \in \Omega$,

$$\begin{aligned} V(K, L_1, \cdots, L_1) &= 2\int_\Omega v(L_1, u)\rho_K(dx) \\ &\leqq 2\int_\Omega v(L_2, u)\rho_K(dx) = V(K, L_2, \cdots, L_2). \end{aligned}$$

On the other hand, suppose $V(K, L_1, \cdots, L_1) \leqq V(K, L_2, \cdots, L_2)$ is valid for all $L_1, L_2 \in \mathcal{K}$ for which we have $v(L_1, u) \leqq v(L_2, u)$, $u \in \Omega$. Let $f \in \mathcal{D}(\Omega)$ be nonnegative and consider $T_K(f) = \rho_f(H_K)$. If we write ρ_f as a difference of measures

$$\rho_f = \rho_2 - \rho_1,\ \rho_1, \rho_2 \in \mathcal{M}_+(\Omega),$$

we get from $f \geqq 0$

$$\int_\Omega |\langle x, u\rangle| (\rho_1 + \lambda)(dx) \leqq \int_\Omega |\langle x, u\rangle| (\rho_2 + \lambda)(dx)$$

for all $u \in \Omega$. But $\rho_1 + \lambda$ and $\rho_2 + \lambda$ are the $(d-1)$-th order surface area measures of bodies $L_1, L_2 \in \mathcal{K}$ ([4, p. 64]).

Hence,

$$\begin{aligned} v(L_1, u) &= \frac{1}{2}\int_\Omega |\langle x, u\rangle| (\rho_1 + \lambda)(dx) \\ &\leqq \frac{1}{2}\int_\Omega |\langle x, u\rangle| (\rho_2 + \lambda)(dx) = v(L_2, u) \text{ for all } u \in \Omega \end{aligned}$$

from which we get, in view of our assumption,

$$V(K, L_1, \cdots, L_1) \leqq V(K, L_2, \cdots, L_2).$$

But,

$$\begin{aligned} 0 &\leqq V(K, L_2, \cdots, L_2) - V(K, L_1, \cdots, L_1) \\ &= \int_\Omega H_K(u)((\rho_2 + \lambda)(du) - (\rho_1 + \lambda)(du)) \\ &= \int_\Omega H_K(u)\rho_f(du) = T_K(f). \end{aligned}$$

Thus, $T_K \in \mathcal{M}_+(\Omega)$ which completes the proof. Q.E.D.

The theorem just proven could be interpreted as follows. The order "$<$" on the set of convex bodies, defined by:

$$K < L, \quad \text{if and only if} \quad v(K, u) \leqq v(L, u) \quad \text{for all} \quad u \in \Omega,$$

is weaker than the inclusion order:

$$K \leqq L, \quad \text{if and only if} \quad K \subset L + x \quad \text{for some} \quad x \in E^d.$$

Mixed volumes of convex bodies are monotone with respect to "$\leqq$".

Monotonicity of certain mixed volumes with respect to the weaker order "$<$" should characterize subclasses of convex bodies. Our monotonicity criterion leads to the class of zonoids.

We have shown in [8], theorem 4, that the monotonicity property of mixed volumes:

$$V(L, K_2, \cdots, K_n) \leqq V(K, K_2, \cdots, K_n), \text{ for convex bodies}$$
$$K, L, K_2, \cdots, K_n \text{ with } L \subset K + x \text{ for some } x \in E^d,$$

can be used in reverse to characterize those pairs L, K of convex bodies which obey $L \subset K + x$ for some $x \in E^d$. This causes us to ask whether, in a similar way, Theorem 5.1 could be inverted. Our final theorem states such an inversion.

THEOREM 5.2. *For convex bodies $L_1, L_2 \subset E^d$, we have $v(L_1, u) \leqq v(L_2, u)$ for all $u \in \Omega$, if and only if*

$$V(K, L_1, \cdots, L_1) \leqq V(K, L_2, \cdots, L_2)$$

for all zonoids $K \in \mathcal{K}$.

PROOF. We have $v(L_1, u) \leqq v(L_2, u)$ for all $u \in \Omega$ if and only if

$$\int_\Omega (v(L_2, u) - v(L_1, u))\rho(du) \geqq 0 \quad \text{for all} \quad \rho \in \mathcal{M}_+(\Omega).$$

Equivalently,

$$\int_\Omega (v(L_2, u) - v(L_1, u))\rho_K(du) \geqq 0 \quad \text{for all zonoids} \quad K \in \mathcal{K}.$$

Then the equation

$$\begin{aligned} 2\int_\Omega & (v(L_2, u) - v(L_1, u))\rho_K(du) \\ &= \int_\Omega \int_\Omega |\langle x, u\rangle| (\mu_{n-1}(L_2; dx) - \mu_{n-1}(L_1; dx))\rho_K(du) \\ &= \int_\Omega H_K(x)(\mu_{n-1}(L_2; dx) - \mu_{n-1}(L_1; dx)) \\ &= V(K, L_2, \cdots, L_2) - V(K, L_1, \cdots, L_1) \end{aligned}$$

proves our statement. Q.E.D.

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ALBERT-LUDWIGS-UNIVERSITÄT
FREIBURG, GERMANY